Fundamental Theorem of Algebra

→ Theory of equations is that branch of Mathematics which deals with the solution of equation.

Theorem: → (7.1) {Factor Theorem}

If α is a root of equation $f(x) = 0$ Then the polynomial $f(x)$ is divisible by $(x-\alpha)$. (i,e; there ~~is~~ will be no ~~remian~~ remainder)

conversely

If the polynomial $f(x)$ be divisible by $(x-\alpha)$ then α is a root of the equation $f(x) = 0$.

Proof: →

⇒

$$x-\alpha)\ f(x)\ (Q(x)$$
$$\overline{\quad R(x) \quad}$$

Let $f(x) = (x-\alpha)\,Q(x) + R(x)$

· Let ~~is~~ α is a root of the equation $f(x) = 0 \Rightarrow f(\alpha) = 0$

When ~~x~~ divide $f(x)$ by $(x-\alpha)$

$$f(x) = (x-\alpha)\,Q(x) + R(x) \quad \text{—} \quad (1)$$

putting $x = \alpha$, we get

$$f(\alpha) = R(\alpha)$$

$$\Rightarrow R(\alpha) = 0$$

~~i.e; f(x) is divisible by~~

i.e; there will be no remainder and as such $f(x)$ is divisible by $(x-\alpha)$.

($\Leftarrow$) Let $f(x)$ is divisible by $(x-\alpha)$ i.e;

$$f(x) = (x-\alpha)\,\phi(x) \qquad \text{——— (2)}$$

putting $x=\alpha$

$$f(\alpha) = 0$$

$\Rightarrow \alpha$ is a root of equation $f(x)=0$.

Remark: $\rightarrow$

Equation - 1st degree - linear
2nd degree - quadratic
3rd degree - cubic
4th degree - bi-quadratic or ~~quad~~ quartic
5th degree - quantic
6th degree - sextic
7th degree - ~~quo~~ quintic.

Every polynomial equation with real coefficients has at least one root

THEOREM: → In an equation with real coefficient, imaginary roots occur in conjugate pairs.

Fundamental Theorem of Algebra.

Statement: → **THEOREM:** → Every equation of nth degree has n roots no more.

Proof: → We consider

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n \quad \text{--(1)}$$

~~Let~~ We assume that (1) has at least one root α_1.

$\therefore$ $f(x)$ is divisible by $(x - \alpha_1)$.

$\therefore$ $f(x) \equiv (x - \alpha) Q_1(x)$,

where $Q_1(x)$ is a ~~degree of~~ polynomial of degree $(n-1)$.

$\therefore$ By the above assumption $Q_1(x) = 0$ also has at least one root α_2.

$\therefore$ $Q_1(x)$ is divisible by $(x - \alpha_2)$.

$\therefore \; \phi_1(x) = (x-\alpha_2)\,\phi_2(x)$

Where $\phi_2(x)$ is a polynomial of degree $(n-2)$.

$\therefore \; f(x) \equiv (x-\alpha_1)(x-\alpha_2)\,\phi_2(x)$

Similarly,

$f(x) \equiv (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\,\phi_3(x)$

and so on lastly

$f(x) \equiv A\,(x-\alpha_1)(x-\alpha_2) \;-\,-\,-\,-\; (x-\alpha_n)$ — (2)

Where A is a constant (independent of x)

Equating co-efficient of x^n from both sides of (2)

We have $a_0 \equiv A$

$\therefore \; f(x) = a_0\,(x-\alpha_1)(x-\alpha_2) \cdots\cdots (x-\alpha_n)$

$\therefore \; f(x) = 0$ if $x = \alpha_1, \alpha_2, \ldots, \alpha_n$

$\therefore \; \alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ are n roots of $f(x)=0$.

Also if α is different from each one of $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ then

$$\alpha - \alpha_1 \neq 0,\ \alpha - \alpha_2 \neq 0, \ldots\ldots\ldots, \alpha - \alpha_n \neq 0.$$

$$\therefore\ f(\alpha) = a_0 (\alpha - \alpha_1)(\alpha - \alpha_2) \ldots\ldots (\alpha - \alpha_n)$$
$$\neq 0 \quad (\because a_0 \neq 0)$$

$\therefore$ α is not a root of $f(x) = 0$

$\therefore$ $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ are only roots of $f(x) = 0$.

proved